On approximate ternary m-derivations and σ-homomorphisms

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Abstract: In this paper we introduce ternary modules over ternary algebras and using fixed point methods, we prove the stability and super-stability of ternary additive, quadratic, cubic and quartic derivations and σ-homomorphisms in such structures for the functional equation

$$f(ax+y)+f(ax-y)=a^{m-2}[f(x+y)+f(x-y)]$$
$$+2(a^2-1)[a^{m-2}f(x)+\frac{(m-2)(1-(m-2)^2)}{6}f(y)].$$

for each $m=1,2,3,4$.

Keywords: Ternary algebras; stability; approximation; derivations; homomorphisms.

1. Introduction

We recall that a nonempty set G is said to be a *ternary groupoid* provided there exists on G a ternary operation $[\cdot,\cdot,\cdot]: G\times G\times G\to G$, which is denoted by $(G,[\cdot,\cdot,\cdot])$. The ternary groupoid $(G,[\cdot,\cdot,\cdot])$ is said to be *commutative* if $[x_1,x_2,x_3]=[x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}]$ for all $x_1,x_2,x_3\in G$ and all permutations σ of $\{1,2,3\}$. If a binary operation $\circ$ is defined on G such that $[x,y,z]=(x\circ y)\circ z$ for all $x,y,z\in G$, then we say that $[\cdot,\cdot,\cdot]$ is derived from $\circ$. We say that $(G,[\cdot,\cdot,\cdot])$ is a *ternary semigroup* if the operation $[\cdot,\cdot,\cdot]$ is associative, i.e., if $[[x,y,z],u,v]=[x,[y,z,u],v]=[x,y,[z,u,v]]$ holds for all $x,y,z,u,v\in G$ (see [1]).

A *ternary Banach algebra* is a complex Banach space A, equipped with a ternary product $(x,y,z)\to[x,y,z]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable and associative in the sense that $[x,y,[z,w,v]]=[x,[w,z,y],v]=[[x,y,z],w,v]$ and satisfies $\|[x,y,z]\|\le\|x\|\cdot\|y\|\cdot\|z\|$.

Consider the functional equation $\Im_1(f)=\Im_2(f)$ $(\Im)$ in a certain general setting. A function g is an approximate solution of $(\Im)$ if $\Im_1(g)$ and $\Im_2(g)$ are close in some sense. The Ulam stability problem asks whether or not there exists a true solution of $(\Im)$ near g. A functional equation is said to be *superstable* if every approximate solution of the equation is an exact solution of the functional equation. The problem of stability of functional equations originated from a question of Ulam [2] concerning the stability of group homomorphisms:

Let $(G_1,*)$ be a group and $(G_2,\star,d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta(\epsilon)>0$ such that, if a mapping $h: G_1\longrightarrow G_2$ satisfies the inequality

$$d(h(x*y),h(x)\star h(y))<\delta$$

for all $x,y\in G_1$, then there exists a homomorphism $H: G_1\to G_2$ with $d(h(x),H(x))<\epsilon$ for all $x\in G_1$?

If the answer is affirmative, we say that the equation of homomorphism $H(x*y)=H(x)\star H(y)$ is *stable*. The concept of stability for a functional equation arises when we replace

the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

In 1941, Hyers [3] gave a first affirmative answer to the question of Ulam for Banach spaces.

Let X and Y be Banach spaces. Assume that $f : X \longrightarrow Y$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in X$ and some $\epsilon > 0$. Then there exists a unique additive mapping $T : X \longrightarrow Y$ such that $\|f(x) - T(x)\| \leq \epsilon$ for all $x \in X$.

A generalized version of the theorem of Hyers for approximately additive mappings was given by Aoki [4] in 1950 (see also [5]). In 1978, a generalized solution for approximately linear mappings was given by Th.M. Rassias [6]. He considered a mapping $f : X \to Y$ satisfying the condition

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$, where $\epsilon \geq 0$ and $0 \leq p < 1$. This result was later extended to all $p \neq 1$ and generalized by Gajda [7], Rassias and Semrl [8], Isac and Rassias [9].

The problem when $p = 1$ is not true. Counterexamples for the corresponding assertion in the case $p = 1$ were constructed by Gadja [7], Rassias and Semrl [8].

On the other hand, Rassias [10, 11] considered the Cauchy difference controlled by a product of different powers of norm. Furthermore, a generalization of Rassias theorems was obtained by Găvruta [12], who replaced

$$\epsilon(\| x \|^p + \| y \|^p)$$

and $\epsilon\|x\|^p\|y\|^p$ by a general control function $\varphi(x, y)$. In 1949 and 1951, Bourgin [13, 14] is the first mathematician dealing with stability of (ring) homomorphism $f(xy) = f(x)f(y)$. The topic of approximation of functional equations on Banach algebras was studied by a number of mathematicians (see [15]–[20]).

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is related to a symmetric bi–additive mapping [21]. It is natural that this equation is called a *quadratic functional equation*. For more details about various results concerning such problems, the readers refer to [22]–[26].

In 2002, Jun and Kim [27] introduced the following cubic functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{1.2}$$

and they established the general solution and the generalized Hyers–Ulam–Rassias stability for the functional equation (1.2). Obviously, the mapping $f(x) = cx^3$ satisfies the functional equation (1.2), which is called the *cubic functional equation*. In 2005, Lee et al. [28] considered the following functional equation

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y) \tag{1.3}$$

It is easy to see that the mapping $f(x) = dx^4$ is a solution of the functional equation (1.3), which is called the *quartic functional equation*.

2. Preliminaries

In 2007, Park [29] investigated the generalized stability of a quadratic mapping $f : A \longrightarrow B$, which is called a C^*-*ternary quadratic mapping* if f is a quadratic mapping satisfies

$$f([x, y, z]) = [f(x), f(y), f(z)] \tag{2.1}$$

for all $x, y, z \in A$. Let A be an algebra. An additive mapping $f : A \to A$ is called a ring derivation if $f(xy) = xf(y) + f(x)y$ holds for all $x, y \in A$. If, in addition, $f(\lambda x) = \lambda f(x)$ for all $x \in A$ and all $\lambda \in \mathbb{F}$, then f is called a linear derivation, where $\mathbb{F}$ denotes the scalar field of A. The stability result concerning derivations between operator algebras was first obtained by Šemrl [30]. In [31], Badora proved the stability of functional equation $f(xy) = xf(y) + f(x)y$, where f is a mapping on normed algebra A with unit.

Recently, shagholi et al.[32] proved the stability of ternary quadratic derivations on ternary Banach algebras. Also Moslehian had investigated the stability and superstability of ternary derivations on C^*-ternary rings [33].

the monomial $f(x) = ax^m$ $(x \in \mathbb{R})$ is a solution of the functional equation (2.2) for each $m = 1, 2, 3, 4$.

$$\begin{aligned} f(ax + y) + f(ax - y) &= a^{m-2}[f(x + y) + f(x - y)] \\ &+ 2(a^2 - 1)[a^{m-2}f(x) + \frac{(m - 2)(1 - (m - 2)^2)}{6}f(y)]. \end{aligned} \tag{2.2}$$

For $m = 1, 2, 3, 4$, the functional equation (2.2) is equivalent to the additive, quadratic, cubic and quartic functional equation, respectively.

The general solution of the functional equation (2.2) for any fixed integers a with $a \neq 0, \pm 1$, was obtained by Eshaghi Gordji et al.[34].

In this paper, we study the further generalized stability of ternary additive, quadratic, cubic and quartic derivations and σ-homomorphisms over ternary Banach algebras via fixed point method for the functional equation (2.2). Moreover, we establish the super-stability of this functional equation by suitable control functions.

Definition 2.1. Let A be a ternary algebra and X be a vector space.

(i) X is a left ternary A-module if mapping $A \times A \times X \to X$ satisfies

(LTM 1) For each fixed $a \in A$, the mapping $x \to [a, b, x]$ is linear on X;
(LTM 2) For each fixed $x \in X$, the mapping $(a, b) \to [a, b, x]$ is bilinear on $A \times A$;
(LTM 3) For all $x \in X$ and $a, b, c, d \in A$, $[a, b, [c, d, x]] = [[a, b, c], d, x] = [a, [b, c, d], x]$.

(ii) X is a middle ternary A-module if mapping $A \times X \times A \to X$ satisfies

(MTM 1) For each fixed $a \in A$, the mapping $x \to [a, x, b]$ is linear on X;
(MTM 2) For each fixed $x \in X$, the mapping $(a, b) \to [a, x, b]$ is bilinear on $A \times A$;
(MTM 3) For all $x \in X$ and $a, b, c, d, e, f \in A$, $[a, [b, [c, x, d], e], f] = [[a, b, c], x, [d, e, f]]$.

(iii) X is a right ternary A-module if mapping $X \times A \times A \to X$ satisfies

(RTM 1) For each fixed $a \in A$, the mapping $x \to [x, a, b]$ is linear on X;
(RTM 2) For each fixed $x \in X$, the mapping $(a, b) \to [x, a, b]$ is bilinear on $A \times A$;
(RTM 3) For all $x \in X$ and $x, y, u, v \in A$, $[[x, a, b], c, d] = [x, [a, b, c], d] = [x, a, [b, c, d]]$.

(iv) X is called ternary A-module if X is left ternary A-module, middle ternary A-module and right ternary A-module and if satisfies in the following condition:

(TM) For any $a, b, c, d, e \in (A \cup X)$, which one of them is in X and the rest are in A, $[[a, b, c], d, e] = [a, [b, c, d], e] = [a, b, [c, d, e]]$.

Definition 2.2. Let A be a normed ternary algebra and X be a vector space.

X is said to be a normed left ternary A-module if X is a left ternary A-module and also satisfies in the following axiom:

(NLTM) $\|[a, b, x]\| \leq \|a\| \cdot \|b\| \cdot \|x\|$ for all $a, b \in A$ and $x \in X$.

Similarly normed middle ternary A-module, normed right ternary A-module and normed ternary A-module are defined. A normed ternary A-module is called Banach ternary A-module if it is complete as a normed linear space.

Definition 2.3. Let A, B are ternary algebra, $H : A \to B$ a function and σ a permutation of $\{1, 2, 3\}$. H is said to be a ternary σ-hommomorphism if for all $a_1, a_2, a_3 \in A$

$$H([a_1, a_2, a_3]) = [H(a_{\sigma(1)}), H(a_{\sigma(2)}), H(a_{\sigma(3)})] \tag{2.3}$$

Definition 2.4. Let A and B be two ternary algebras.

(1) A mapping $f : A \to B$ is called a *ternary additive σ-homomorphism* (briefly, ternary 1-σ-homomorphism) if f is an additive mapping satisfying (2.3) for all $a_1, a_2, a_3 \in A$.

(2) A mapping $f : A \to B$ is called a *ternary quadratic σ-homomorphism* (briefly, ternary 2-σ-homomorphism) if f is a quadratic mapping satisfying (2.3) for all $a_1, a_2, a_3 \in A$.

(3) A mapping $f : A \to B$ is called a *ternary cubic σ-homomorphism* (briefly, ternary 3-σ-homomorphism) if f is a cubic mapping satisfying (2.3) for all $a_1, a_2, a_3 \in A$.

(4) A mapping $f : A \to B$ is called a *ternary quartic σ-homomorphism* (briefly, ternary 4-σ-homomorphism) if f is a quartic mapping satisfying (2.3) for all $a_1, a_2, a_3 \in A$.

Definition 2.5. Let A be a ternary algebra and let X be a Banach ternary A-module.

(1) A mapping $f : A \to X$ is called a *ternary additive derivation* (briefly, ternary 1-derivation) if f is an additive mapping that satisfies
$f([x, y, z]) = [f(x), y, z] + [x, f(y), z] + [x, y, f(z)]$ for all $x, y, z \in A$.

(2) A mapping $f : A \to X$ is called a *ternary quadratic derivation* (briefly, ternary 2-derivation) if f is a quadratic mapping that satisfies
$f([x, y, z]) = [f(x), y^2, z^2] + [x^2, f(y), z^2] + [x^2, y^2, f(z)]$ for all $x, y, z \in A$.

(3) A mapping $f : A \to X$ is called a *ternary cubic derivation* (briefly, ternary 3-derivation) if f is a cubic mapping that satisfies
$f([x, y, z]) = [f(x), y^3, z^3] + [x^3, f(y), z^3] + [x^3, y^3, f(z)]$ for all $x, y, z \in A$.

(4) A mapping $f : A \to X$ is called a *ternary quartic derivation* (briefly, ternary 4-derivation) if f is a quartic mapping that satisfies
$f([x, y, z]) = [f(x), y^4, z^4] + [x^4, f(y), z^4] + [x^4, y^4, f(z)]$ for all $x, y, z \in A$.

Now, we state the following notion of fixed point theorem. For the proof, refer to Chapter 5 in [35] and [36, 37]. In 2003, Cădariu and Radu [38] proposed a new method for obtaining

the existence of exact solutions and error estimations, based on the fixed point alternative (see also [39]–[41]).

Let (X, d) be a generalized metric space. We say that a mapping $T : X \to X$ satisfies a Lipschitz condition if there exists a constant $L \geq 0$ such that $d(Tx, Ty) \leq Ld(x, y)$ for all $x, y \in X$, where the number L is called the Lipschitz constant. If the Lipschitz constant L is less than 1, then the mapping T is called a *strictly contractive mapping*. Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity.

The following theorem is famous to fixed point theorm.

Theorem 2.6. *Suppose that (Ω, d) is a complete generalized metric space and $T : \Omega \to \Omega$ is a strictly contractive mapping with the Lipschitz constant L. Then, for any $x \in \Omega$, either*

$$d(T^m x, T^{m+1} x) = \infty, \quad \forall m \geq 0,$$

or there exists a natural number m_0 such that
(1) $d(T^m x, T^{m+1} x) < \infty$ *for all* $m \geq m_0$*;*
(2) *the sequence $\{T^m x\}$ is convergent to a fixed point y^* of T;*
(3) *y^* is the unique fixed point of T in $\Lambda = \{y \in \Omega : d(T^{m_0} x, y) < \infty\}$;*
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ *for all* $y \in \Lambda$.

3. Approximation of ternary m-derivation between ternary algebras

In this section, we investigate the generalized stability of ternary m-derivation between ternary Banach algebras for the functional equation (2.2).

Throughout this section, we suppose that A is ternary Banach algebra and X is Banach ternary A-module. For convenience, we use the following abbreviation: for any function $f : A \to X$,

$$\begin{aligned}\Delta_m f(x, y) &= f(ax + y) + f(ax - y) - a^{m-2}[f(x + y) + f(x - y)] \\ &\quad - 2(a^2 - 1)[a^{m-2} f(x) + \frac{(m-2)(1-(m-2)^2)}{6} f(y)]\end{aligned}$$

for all $x, y \in X$ and any fixed integers a with $a \neq 0, \pm 1$.

From now on, let m be a positive integer less than 5.

Theorem 3.1. *Let $f : A \to X$ be a mapping for which there exist functions $\varphi_m : A \times A \to [0, \infty)$ and $\psi_m : A \times A \times A \to [0, \infty)$ such that*

$$\|\Delta_m f(x, y)\| \leq \varphi_m(x, y), \tag{3.1}$$

$$\|f([x, y, z]) - [f(x), y^m, z^m] - [x^m, f(y), z^m] - [x^m, y^m, f(z)]\| \leq \psi_m(x, y, z) \tag{3.2}$$

for all $x, y, z \in A$. If there exists a constant $0 < L < 1$ such that

$$\varphi_m\Big(\frac{x}{a}, \frac{y}{a}\Big) \leq \frac{L}{|a|^m} \varphi_m(x, y), \tag{3.3}$$

$$\psi_m\Big(\frac{x}{a}, \frac{y}{a}, \frac{z}{a}\Big) \leq \frac{L}{|a|^{3m}} \psi_m(x, y, z) \tag{3.4}$$

for all $x, y, z \in A$, then there exists a unique ternary m-derivation $\mathfrak{F} : A \to X$ such that

$$\|f(x) - \mathfrak{F}(x)\| \leq \frac{L}{2|a|^m(1-L)} \varphi_m(x, 0) \tag{3.5}$$

for all $x \in A$.

Proof. It follows from (3.3) and (3.4) that

$$\lim_{n\to\infty}|a|^{mn}\varphi_m\Big(\frac{x}{a^n},\frac{y}{a^n}\Big)=0, \tag{3.6}$$

$$\lim_{n\to\infty}|a|^{3mn}\psi_m\Big(\frac{x}{a^n},\frac{y}{a^n},\frac{z}{z^n}\Big)=0 \tag{3.7}$$

for all $x,y,z\in X$. By (3.6), $\lim_{n\to\infty}|a|^{mn}\varphi_m(0,0)=0$ and so $\varphi_m(0,0)=0$. Letting $x=y=0$ in (3.1), we get $\|f(0)\|\le\varphi_m(0,0)=0$ and so $f(0)=0$.

Let $\Omega=\{g:g:A\to X,\ g(0)=0\}$. We introduce a generalized metric on Ω as follows:

$$d(g,h)=d_{\varphi_m}(g,h)=\inf\{K\in(0,\infty):\ \|g(x)-h(x)\|\le K\varphi_m(x,0),\ \forall x\in A\}.$$

It is easy to show that (Ω,d) is a generalized complete metric space [39].

Now, we consider the mapping $T:\Omega\to\Omega$ defined by $Tg(x)=a^m\ g(\frac{x}{a})$ for all $x\in A$ and $g\in\Omega$. Note that, for all $g,h\in\Omega$ and $x\in A$,

$$\begin{aligned} d(g,h)<K\ &\Rightarrow\ \|g(x)-h(x)\|\le K\varphi_m(x,0)\\ &\Rightarrow\ \|a^mg(\frac{x}{a})-a^mh(\frac{x}{a})\|\le|a|^m\ K\ \varphi_m(\frac{x}{a},0)\\ &\Rightarrow\ \|a^mg(\frac{x}{a})-a^mh(\frac{x}{a})\|\le L\ K\ \varphi_m(x,0)\\ &\Rightarrow\ d(Tg,Th)\le L\ K.\end{aligned}$$

Hence we see that

$$d(Tg,Th)\le L\ d(g,h)$$

for all $g,h\in\Omega$, that is, T is a strictly self-mapping of Ω with the Lipschitz constant L. Putting $y=0$ in (3.1), we have

$$\|2f(ax)-2a^mf(x)\|\le\varphi_m(x,0) \tag{3.8}$$

for all $x\in A$ and so

$$\Big\|f(x)-a^mf\Big(\frac{x}{a}\Big)\Big\|\le\frac{1}{2}\varphi_m\Big(\frac{x}{a},0\Big)\le\frac{L}{2|a|^m}\ \varphi_m(x,0)$$

for all $x\in A$, that is, $d(f,Tf)\le\frac{L}{2|a|^m}<\infty$.

Now, from Theorem 2.5, it follows that there exists a fixed point $\mathfrak{F}$ of T in Ω such that

$$\mathfrak{F}(x)=\lim_{n\to\infty}a^{mn}f\Big(\frac{x}{a^n}\Big) \tag{3.9}$$

for all $x\in A$ since $\lim_{n\to\infty}d(T^nf,\mathfrak{F})=0$.

On the other hand, it follows from (3.1), (3.6) and (3.9) that

$$\|\Delta_m\mathfrak{F}(x,y)\|=\lim_{n\to\infty}|a|^{mn}\Big\|\Delta_mf\Big(\frac{x}{a^n},\frac{y}{a^n}\Big)\Big\|\le\lim_{n\to\infty}|a|^{mn}\varphi_m\Big(\frac{x}{a^n},\frac{y}{a^n}\Big)=0$$

for all $x,y\in A$ and so $\Delta_m\mathfrak{F}(x,y)=0$. By the result in [34], $\mathfrak{F}$ is $m-$mapping and so it follows from the definition of $\mathfrak{F}$, (3.2) and (3.7) that

$$\begin{aligned}&\|\mathfrak{F}([x,y,z])-[\mathfrak{F}(x),y^m,z^m]-[x^m,\mathfrak{F}(y),z^m]-[x^m,y^m,\mathfrak{F}(z))]\|\\ &=\lim_{n\to\infty}|a|^{3mn}\Big\|f\Big(\frac{[x,y,z]}{a^{3n}}\Big)-\Big[f\Big(\frac{x}{a^n}\Big),\Big(\frac{y}{a^n}\Big)^m,\Big(\frac{z}{a^n}\Big)^m\Big]-\Big[\Big(\frac{x}{a^n}\Big)^m,f\Big(\frac{y}{a^n}\Big),\Big(\frac{z}{a^n}\Big)^m\Big]\\ &-\Big[\Big(\frac{x}{a^n}\Big)^m,\Big(\frac{y}{a^n}\Big)^m,f\Big(\frac{z}{a^n}\Big)\Big]\Big\|\le\lim_{n\to\infty}|a|^{3mn}\psi_m\Big(\frac{x}{a^n},\frac{y}{a^n},\frac{z}{a^n}\Big)=0\end{aligned}$$

for all $x,y,z\in A$ and so

$$\mathfrak{F}([x,y,z])=[\mathfrak{F}(x),y^m,z^m]+[x^m,\mathfrak{F}(y),z^m]+[x^m,y^m,\mathfrak{F}(z))].$$

According to Theorem 2.2, since $\mathfrak{F}$ is the unique fixed point of T in the set $\Lambda = \{g \in \Omega : d(f,g) < \infty\}$, $\mathfrak{F}$ is the unique mapping such that

$$\|f(x) - \mathfrak{F}(x)\| \leq K\ \varphi_m(x,0)$$

for all $x \in A$ and $K > 0$. Again, using Theorem 2.2, we have

$$d(f,\mathfrak{F}) \leq \frac{1}{1-L}d(f,Tf) \leq \frac{L}{2|a|^m(1-L)}$$

and so

$$\|f(x) - \mathfrak{F}(x)\| \leq \frac{L}{2|a|^m(1-L)}\ \varphi_m(x,0)$$

for all $x \in A$. This completes the proof. □

Corollary 3.2. *Let θ, r, p be non–negative real numbers with $r, p > m$ and $\frac{3p-r}{2} \geq m$. Suppose that $f : A \to X$ is a mapping such that*

$$\|\Delta_m f(x,y)\| \leq \theta(\|x\|^r + \|y\|^r), \tag{3.10}$$

$$\|f([x,y,z]) - [f(x), y^m, z^m] - [x^m, f(y), z^m] - [x^m, y^m, f(z)]\| \leq \theta(\|x\|^p.\|y\|^p.\|z\|^p) \tag{3.11}$$

for all $x, y, z \in A$. Then there exists a unique ternary m-derivation $\mathfrak{F} : A \to X$ satisfying

$$\|f(x) - \mathfrak{F}(x)\| \leq \frac{\theta}{2(|a|^r - |a|^m)}\|x\|^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.1 by taking

$$\varphi_m(x,y) := \theta(\|x\|^r + \|y\|^r), \quad \psi_m(x,y,z) := \theta(\|x\|^p \cdot \|y\|^p \cdot \|z\|^p)$$

for all $x, y, z \in A$. Then we can choose $L = |a|^{m-r}$ and so the desired conclusion follows. □

Remark 3.3. Let $f : A \to X$ be a mapping with $f(0) = 0$ such that there exist functions $\varphi_m : A \times A \to [0,\infty)$ and $\psi_m : A \times A \times A \to [0,\infty)$ satisfying (3.1) and (3.2). Let $0 < L < 1$ be a constant such that

$$\varphi_m(ax,ay) \leq |a|^m L\varphi_m(x,y), \quad \psi_m(ax,ay,az) \leq |a|^{3m} L\psi_m(x,y,z)$$

for all $x, y, z \in A$. By the similar method as in the proof of Theorem 3.1, one can show that there exists a unique ternary m-derivation $\mathfrak{F} : A \to X$ satisfying

$$\|f(x) - \mathfrak{F}(x)\| \leq \frac{1}{2|a|^m(1-L)}\varphi_m(x,0)$$

for all $x \in A$. For the case

$$\varphi_m(x,y) := \delta + \theta(\|x\|^r + \|y\|^r), \quad \psi_m(x,y,z) := \delta + \theta(\|x\|^p \cdot \|y\|^p \cdot \|z\|^p),$$

where θ, δ are non–negative real numbers and $0 < r, p < m$ and $\frac{3p-r}{2} \leq m$, there exists a unique ternary m-derivation $\mathfrak{F} : A \to X$ satisfying

$$\|f(x) - \mathfrak{F}(x)\| \leq \frac{\delta}{2(|a|^m - |a|^r)} + \frac{\theta}{2(|a|^m - |a|^r)}\|x\|^r$$

for all $x \in A$.

In the following, we formulate and prove a theorem in super-stability of ternary m-derivation in ternary algebras for the functional equation (2.2).

Theorem 3.4. *Suppose that there exist functions $\varphi_m : A \times A \to [0,\infty)$, $\psi_m : A \times A \times A \to [0,\infty)$ and a constant $0 < L < 1$ such that*

$$\varphi_m\Big(0, \frac{y}{a}\Big) \leq \frac{L}{|a|^m}\varphi_m(0,y), \tag{3.12}$$

$$\psi_m\Big(\frac{x}{a}, \frac{y}{a}, \frac{z}{a}\Big) \leq \frac{L}{|a|^{3m}}\psi_m(x,y,z) \tag{3.13}$$

for all $x, y, z \in A$. Moreover, if $f : A \to X$ is a mapping such that

$$\|\Delta_m f(x,y)\| \leq \varphi_m(0,y), \tag{3.14}$$

$$\|f([x,y,z]) - [f(x), y^m, z^m] - [x^m, f(y), z^m] - [x^m, y^m, f(z)]\| \leq \psi_m(x,y,z) \tag{3.15}$$

for all $x, y, z \in A$, then f is a ternary m-derivation.

Proof. It follows from (3.12) and (3.13) that

$$\lim_{n\to\infty} |a|^{mn}\varphi_m\Big(0, \frac{y}{a^n}\Big) = 0, \tag{3.16}$$

$$\lim_{n\to\infty} |a|^{3mn}\psi_m\Big(\frac{x}{a^n}, \frac{y}{a^n}, \frac{z}{a^n}\Big) = 0 \tag{3.17}$$

for all $x, y, z \in A$. We have $f(0) = 0$ since $\varphi_m(0,0) = 0$. Letting $y = 0$ in (3.14), we get $f(ax) = a^m f(x)$ for all $x \in A$. By using induction, we obtain

$$f(a^n x) = a^{mn} f(x)$$

for all $x \in A$ and $n \in \mathbb{N}$ and so

$$f(x) = a^{mn} f\Big(\frac{x}{a^n}\Big) \tag{3.18}$$

for all $x \in A$ and $n \in \mathbb{N}$. It follows from (3.15) and (3.18) that

$$\begin{aligned} &\|f([x,y,z]) - [f(x), y^m, z^m] - [x^m, f(y), z^m] - [x^m, y^m, f(z)]\| \\ &= |a|^{3mn}\Big\|f\Big(\frac{[x,y,z]}{a^{3n}}\Big) - \Big[f\Big(\frac{x}{a^n}\Big), \Big(\frac{y}{a^n}\Big)^m, \Big(\frac{z}{a^n}\Big)^m\Big] - \Big[\Big(\frac{x}{a^n}\Big)^m, f\Big(\frac{y}{a^n}\Big), \Big(\frac{z}{a^n}\Big)^m\Big] \\ &- \Big[\Big(\frac{x}{a^n}\Big)^m, \Big(\frac{y}{a^n}\Big)^m, f\Big(\frac{z}{a^n}\Big)\Big]\Big\| \leq |a|^{3mn}\psi_m\Big(\frac{x}{a^n}, \frac{y}{a^n}, \frac{z}{a^n}\Big) \end{aligned} \tag{3.19}$$

for all $x, y, z \in A$ and $n \in \mathbb{N}$. Hence, letting $n \to \infty$ in (3.19) and using (3.17), we have $f([x,y,z]) = [f(x), y^m, z^m] + [x^m, f(y), z^m] + [x^m, y^m, f(z)]$ for all $x, y, z \in A$.

On the other hand, we have

$$\|\Delta_m f(x,y)\| = |a|^{mn}\Big\|\Delta_m f\Big(\frac{x}{a^n}, \frac{y}{a^n}\Big)\Big\| \leq |a|^{mn}\varphi_m\Big(0, \frac{y}{a^n}\Big) \tag{3.20}$$

for all $x, y \in A$ and $n \in \mathbb{N}$. Thus, letting $n \to \infty$ in (3.20) and using (3.16), we have $\Delta_m f(x,y) = 0$ for all $x, y \in A$. Therefore, f is a ternary m-derivation. This completes the proof. $\square$

Corollary 3.5. *Let θ, r, s be non–negative real numbers with $r > m$ and $s > 3m$. If $f : A \to X$ is a function such that*

$$\|\Delta_m f(x,y)\| \leq \theta\|y\|^r,$$

$$\|f([x,y,z]) - [f(x), y^m, z^m] - [x^m, f(y), z^m] - [x^m, y^m, f(z)]\| \leq \theta(\|x\|^s + \|y\|^s + \|z\|^s)$$

for all $x, y, z \in A$, then f is a ternary m-derivation.

Remark 3.6. Let θ, r be non–negative real numbers with $r < m$. Suppose that there exists a function $\psi_m : A \times A \times A \to [0, \infty)$ and a constant $0 < L < 1$ such that

$$\psi_m(ax, ay, az) \leq |a|^{3m} L \psi_m(x, y, z)$$

for all $x, y, z \in A$. Moreover, if $f : A \to X$ is a mapping such that

$$\|\Delta_m f(x, y)\| \leq \theta \|y\|^r,$$

$$\|f([x, y, z]) - [f(x), y^m, z^m] - [x^m, f(y), z^m] - [x^m, y^m, f(z)]\| \leq \psi_m(x, y, z)$$

for all $x, y, z \in A$, then f is a ternary m-derivation.

4. Approximation of ternary m-σ-Homomorphism between ternary algebras

In this section, we investigate the generalized stability of ternary m-σ-Homomorphism between ternary Banach algebras for the functional equation (2.2).

Throughout this section, we suppose that A, B are two ternary Banach algebra. From now on, let m be a positive integer less than 5 and σ a permutation of $\{1, 2, 3\}$.

Theorem 4.1. *Let $f : A \to B$ be a mapping for which there exist functions $\varphi_m : A \times A \to [0, \infty)$ and $\psi_m : A \times A \times A \to [0, \infty)$ such that*

$$\|\Delta_m f(x_1, x_2)\| \leq \varphi_m(x_1, x_2), \tag{4.1}$$

$$\|f([x_1, x_2, x_3]) - [f(x_{\sigma(1)}), f(x_{\sigma(2)}), f(x_{\sigma(3)})]\| \leq \psi_m(x_1, x_2, x_3) \tag{4.2}$$

for all $x_1, x_2, x_3 \in A$. If there exists a constant $0 < L < 1$ such that

$$\varphi_m\Big(\frac{x_1}{a}, \frac{x_2}{a}\Big) \leq \frac{L}{|a|^m} \varphi_m(x_1, x_2), \tag{4.3}$$

$$\psi_m\Big(\frac{x_1}{a}, \frac{x_2}{a}, \frac{x_3}{a}\Big) \leq \frac{L}{|a|^{3m}} \psi_m(x_1, x_2, x_3) \tag{4.4}$$

for all $x_1, x_2, x_3 \in A$ then there exists a unique ternary m-σ-hommomorphism $H : A \to B$ such that

$$\|f(x_1) - H(x_1)\| \leq \frac{L}{2|a|^m(1 - L)} \varphi_m(x_1, 0) \tag{4.5}$$

for all $x_1 \in A$.

Proof. Let us define Ω, d and $T : \Omega \to \Omega$ by the same definitions as in the proof of Theorem 3.1, one can show that T has a unique fixed point H in Ω such that $H(x_1) = \lim_{n\to\infty} a^{mn} f\Big(\frac{x_1}{a^n}\Big)$ that

$$\|\Delta_m H(x_1, x_2)\| = \lim_{n\to\infty} |a|^{mn} \Big\|\Delta_m f\Big(\frac{x_1}{a^n}, \frac{x_2}{a^n}\Big)\Big\| \leq \lim_{n\to\infty} |a|^{mn} \varphi_m\Big(\frac{x_1}{a^n}, \frac{x_2}{a^n}\Big) = 0$$

for all $x_1, x_2 \in A$ and so $\Delta_m H(x_1, x_2) = 0$. By the result in [34], H is $m-$mapping. On the other hand, it follows from the definition of H that

$$\begin{aligned}
&\|H([x_1, x_2, x_3]) - [H(x_{\sigma(1)}), H(x_{\sigma(2)}), H(x_{\sigma(3)})]\| \\
&= \lim_{n\to\infty} |a|^{3mn} \Big\| f\Big(\frac{[x_1, x_2, x_3]}{a^{3n}}\Big) - \Big[f\Big(\frac{x_{\sigma(1)}}{a^n}\Big), f\Big(\frac{x_{\sigma(2)}}{a^n}\Big), f\Big(\frac{x_{\sigma(3)}}{a^n}\Big)\Big]\Big\| \\
&\leq \lim_{n\to\infty} |a|^{3mn} \psi_m\Big(\frac{x_1}{a^n}, \frac{x_2}{a^n}, \frac{x_3}{a^n}\Big) = 0
\end{aligned}$$

for all $x_1, x_2, x_3 \in A$ and so

$H([x_1,x_2,x_3]) = [H(x_{\sigma(1)}), H(x_{\sigma(2)}), H(x_{\sigma(3)})]$.

According to Theorem 2.5, since H is the unique fixed point of T in the set $\Lambda = \{g \in \Omega : d(f,g) < \infty\}$, H is the unique mapping such that

$$\|f(x_1) - H(x_1)\| \le K \ \varphi_m(x_1, 0)$$

for all $x_1 \in A$ and $K > 0$. Again, using Theorem 2.5, we have

$$d(f,H) \le \frac{1}{1-L} d(f,Tf) \le \frac{L}{2|a|^m(1-L)}$$

and so

$$\|f(x_1) - H(x_1)\| \le \frac{L}{2|a|^m(1-L)} \ \varphi_m(x_1, 0)$$

for all $x_1 \in A$. This completes the proof. $\square$

Corollary 4.2. *Let θ, r, p be non–negative real numbers with $r, p > m$ and $\frac{3p-r}{2} \ge m$. Suppose that $f : A \to B$ is a mapping such that*

$$\|\Delta_m f(x_1,x_2)\| \le \theta(\|x_1\|^r + \|x_2\|^r), \tag{4.6}$$

$$\|f([x_1,x_2,x_3]) - [f(x_{\sigma(1)}), f(x_{\sigma(2)}), f(x_{\sigma(3)})]\| \le \theta(\|x_1\|^p.\|x_2\|^p.\|x_3\|^p) \tag{4.7}$$

for all $x_1, x_2, x_3 \in A$. Then there exists a unique ternary m-σ-homomorphism $H : A \to B$ satisfying

$$\|f(x_1) - H(x_1)\| \le \frac{\theta}{2(|a|^r - |a|^m)} \|x_1\|^r$$

for all $x_1 \in A$.

Proof. The proof follows from Theorem 4.1 by taking

$$\varphi_m(x_1,x_2) := \theta(\|x_1\|^r + \|x_2\|^r), \quad \psi_m(x_1,x_2,x_3) := \theta(\|x_1\|^p \cdot \|x_2\|^p \cdot \|x_3\|^p)$$

for all $x_1, x_2, x_3 \in A$. Then we can choose $L = |a|^{m-r}$ and so the desired conclusion follows. $\square$

Remark 4.3. Let $f : A \to B$ be a mapping with $f(0) = 0$ such that there exist functions $\varphi_m : A \times A \to [0,\infty)$ and $\psi_m : A \times A \times A \to [0,\infty)$ satisfying (4.1) and (4.2). Let $0 < L < 1$ be a constant such that

$$\varphi_m(ax_1, ax_2) \le |a|^m L \varphi_m(x_1,x_2), \quad \psi_m(ax_1,ax_2,ax_3) \le |a|^{3m} L \psi_m(x_1,x_2,x_3)$$

for all $x_1, x_2, x_3 \in A$. By the similar method as in the proof of Theorem 4.1, one can show that there exists a unique ternary m-σ-homomorphism $H : A \to B$ satisfying

$$\|f(x_1) - H(x_1)\| \le \frac{1}{2|a|^m(1-L)} \varphi_m(x_1, 0)$$

for all $x_1 \in A$. For the case

$$\varphi_m(x_1,x_2) := \theta(\|x_1\|^r + \|x_2\|^r), \quad \psi_m(x_1,x_2,x_3) := \theta(\|x_1\|^p + \|x_2\|^p + \|x_3\|^p),$$

where θ is non–negative real numbers and $0 < r < m$, $0 < p < 3m$ and $\frac{p-r}{2} \le m$, there exists a unique ternary m-σ-homomorphism $H : A \to B$ satisfying

$$\|f(x_1) - H(x_1)\| \le \frac{\theta}{2(|a|^m - |a|^r)} \|x_1\|^r$$

for all $x_1 \in A$.

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